An Analytic Solution for Constant-Thrust, Optimal-Coast, Minimum-Propellant Space Trajectories

A. L. KORNHAUSER*
University of Minnesota, Minneapolis, Minn.

P. M. LION†
Princeton University, Princeton, N. J.

G. A. HAZELRIGG‡
Convair Division of General Dynamics, San Diego, Calif.

The variational formulation of the minimum-propellant control of a space vehicle in an inverse-square gravitational field leads to a nonlinear, two-point, boundary-value problem. Although for the case of unbounded thrust-magnitude (impulsive propulsion) solutions are easily found, numerical solutions for the case of bounded thrust are not so readily obtainable. This paper gives an analytic method for determining the optimal control of a thrust-limited, rocket-powered vehicle in terms of an expansion about the optimal impulsive solution. The expansion is in terms of two independent vehicle parameters, the initial thrust-acceleration and the rocket jet exhaust velocity (a measure of the specific impulse). The resulting solution is valid over a wide range of vehicle parameters of interest and gives the thrust-vector-time history on the optimal bounded-thrust trajectory. The solution is easy to implement on a high-speed digital computer and requires very little computing time.

Introduction

THE problem of minimum-propellant spaceflight of a rocket-powered vehicle has been approached by many researchers using variational techniques such as the calculus of variations or Pontryagin's maximum principle. The inevitable result is a two-point, boundary-value problem which, in most interesting cases, is also nonlinear. Classically, solutions have been obtained by numerical methods involving trajectory integration and end-point iteration. These methods are cumbersome and require excessive running times on a high-speed digital computer.

This paper develops an analytic solution of minimum-propellant space trajectories with bounded thrust in a central-body gravitational field. The solution is an extension to the bounded-thrust case of the work presented in Ref. 1. Its principal result is an analytic expression for the minimum-propellant control of a bounded-thrust spacecraft; i.e., the thrusting- and thrust pointing-time histories. The solution is obtained in series form as an expansion about the optimal impulsive (unbounded thrust-magnitude) trajectory satisfying the same boundary conditions.

The two propulsion parameters which affect the trajectory are a_0, the initial thrust-acceleration and v_j, the jet exhaust velocity of the rocket. The solution is sought in the form of a series in $1/a_0$ and $1/v_j$ constructed so that as $1/a_0 \rightarrow 0$, the solution approaches the impulsive solution, and as $1/v_j \rightarrow 0$, the solution approaches the constant thrust-acceleration case of Ref. 1. Clearly, this technique is most accurate for high thrust levels since the expansion is about the impulsive case. However, numerical experience to date has indicated that the approximation works well into the range of propulsion parameters appropriate to electric propulsion. In the following analysis it is assumed that the optimal impulsive trajectory and its adjoint are known.

Received April 1, 1970; revision received November 5, 1970.

* Assistant Professor, Department of Aerospace Engineering and Mechanics. Member AIAA.
† Associate Professor, Department of Aerospace and Mechanical Sciences. Member AIAA.
‡ Staff Scientist, Trajectory Analysis. Member AIAA.

Equations of Motion and Necessary Conditions

The equations of spacecraft motion can be written

$$\dot{x} = f(x) + u; \quad \dot{m} = -T/v_j \tag{1}$$

where the six vectors $x' = (r',v')$, $u' = (T/m)(o',\eta')$, $f = (v', g(r)')$ are conveniently partitioned into three vectors as shown: r denotes position, v velocity, g gravitational acceleration; η is a unit vector in the direction of the thrust, T is the thrust magnitude, m the mass where $m(0) = m_0 = 1$,0 is a null vector, and $'$ a transpose. Since we are dealing with bounded thrust, we have

$$0 \leq T \leq a_0 m_0$$

The adjoint vector on position and velocity ψ is given by

$$\dot{\psi} = -F'\psi \tag{2}$$

where $F = \partial f/\partial x$. This (6×6) matrix is partitioned into (3×3) blocks as follows

$$F = \begin{bmatrix} 0 & I \\ G & 0 \end{bmatrix}$$

where $G = \partial g/\partial r$ and I is a (3×3) identity matrix. Similarly, it is convenient to partition ψ into two three vectors, $\psi' = (-\dot{\lambda}',\lambda')$ where λ, the adjoint vector associated with velocity, is known as the "primer vector." The adjoint variable associated with mass is denoted by σ. σ increases monotonically according to the relation

$$\dot{\sigma} = (T/m^2)p \tag{3}$$

where $p = |\lambda|$.

The minimum-propellant trajectory is defined as that trajectory which satisfies Eqs. (1) and the boundary conditions

(for rendezvous trajectories)

$$x(0) = x_0 \qquad x(t_f) = x_f$$

and maximizes $m(t_f)$.

In terms of the switching function

$$s = p - \sigma m/v_j \tag{4}$$

the necessary conditions which must hold on an optimal trajectory can be stated as follows:

$$\begin{aligned} s > 0 \qquad T &= a_0 m_0 \qquad \eta = \lambda/p \\ s < 0 \qquad T &= 0 \end{aligned} \tag{5}$$

These conditions reflect the well-known result that the thrust-magnitude alternates between periods of maximum thrust and no thrust and is aligned with the primer vector during thrust periods. (Singular arcs where $s = 0$ for a finite time are not considered in this analysis.)

During thrusting periods, conditions (5) lead to the result

$$u(t) = a_0[m(t)p(t)]^{-1}M\psi(t) \tag{6}$$

where M is the (6×6) matrix

$$M = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$$

Figure 1 shows a plot of p and $\sigma m/v_j$ vs t. Where these curves intersect, s is zero and thrust switches occur. A corresponding thrust program is also shown.

In the case where the upper bound on thrust is removed $(a_0 \to \infty)$, the optimal control becomes a sequence of velocity impulses

$$\sum_{k=1}^{n} \Delta \mathbf{v}_k \, \delta(t - t_k)$$

where δ indicates the Dirac-delta function, t_k is the time of the kth "impulse point" and n is the total number of impulses.

Necessary conditions for an optimal impulsive trajectory can be stated in terms of the primer vector.[6] Thus, for the case of impulsive thrust, Eq. (5) can be written in the form

$$u^{(0)}(t) = \Delta v_k \delta(t - t_k) M \psi^{(0)}(t) \tag{7}$$

where Δv_k is the magnitude of the kth impulse. Figure 1 also shows the optimal thrust program for the impulsive case.

Expansion about the Impulsive Trajectory

The following analysis is based on the assumptions that: a) the optimal finite-thrust trajectory approaches the optimal impulsive trajectory as $a_0 \to \infty$, b) the optimal bounded-thrust trajectory approaches the optimal bounded-thrust-acceleration trajectory as $v_j \to \infty$, and c) the optimal impulsive trajectory is known for the prescribed initial and final conditions. The state and adjoint vectors on the optimal impulsive trajectory are denoted by $x^{(0)}$ and $\psi^{(0)}$, respectively. Based on the preceding assumptions, the state and adjoint vectors are expanded about the impulsive case as series in $\epsilon = 1/a_0$ and $\delta = 1/v_j$:

$$\begin{aligned} x &= x^{(0)} + \epsilon x^{(1,0)} + \epsilon\delta x^{(1,1)} + \epsilon^2 x^{(2,0)} + \ldots \\ \psi &= \psi^{(0)} + \epsilon\psi^{(1,0)} + \epsilon\delta\psi^{(1,1)} + \epsilon^2\psi^{(2,0)} + \ldots \end{aligned} \tag{8}$$

where each coefficient of the above power series is assumed to be of the first order. Note that because of assumption a, Eqs. (8) contain no terms of the form $\delta^n, n = 1, 2, 3, \ldots$.

Using the equations of motion, recursion relationships are sought in order to calculate the higher order corrections. Most of the difficulty arises from the additional requirement that the optimality conditions (5) be satisfied to each order.

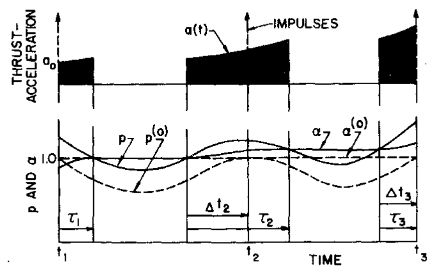

Fig. 1 Switch-point analysis and corresponding thrust-acceleration-time history.

For this reason, it is also necessary to expand the thrust intervals on the finite-thrust trajectory in a series:

$$\tau_k = \epsilon\tau^{(1,0)} + \epsilon^2\tau^{(2,0)} + \epsilon\delta\tau^{(1,1)} + \ldots \tag{9}$$

where τ_k is the length of the kth thrust interval. Note that as $a_0 \to \infty$, $\tau_k \to 0$. Subsequent analysis shows that $\tau_k^{(0)} = \Delta v_k$ the kth velocity impulse. Therefore, as $a_0 \to \infty$, $a_0\tau_k \to \Delta v_k$.

Define $\Delta x(t) = x(t) - x^{(0)}(t)$, $\Delta\psi(t) = \psi(t) - \psi^{(0)}(t)$ and $\Delta u(t) = u(t) - u^{(0)}(t)$ as the contemporaneous variations in state, adjoint, and control. They satisfy the following linear equations to all order:

$$\Delta\dot{x} = F\Delta x + \Delta u \tag{10}$$

$$\Delta\dot{\psi} = -F'\Delta\psi \tag{11}$$

For nonlinear gravitational fields Eq. (10) is valid only through first order since the second-order term $\frac{1}{2}\Delta x^T(\partial^2 g/\partial x^2)\Delta x$, $[O(\epsilon^2)]$, has been neglected. Equation (11) is not valid since the first-order term ψ'. $(\partial^2 g/\partial x^2)\,\Delta x$, $[O(\epsilon)]$, has been neglected. For gravitational field having $\partial^2 g/\partial x^2 = O(\epsilon)$ (linear gravitational fields), Eq. (10) is then valid through second order and Eq. (11) is valid through first order. The solution to Eq. (10) is

$$\Delta x(t) = \int_0^t \Phi(t,\xi)\Delta u(\xi)d\xi + \Phi(t,t_0)\Delta x(0) \tag{12}$$

where $\Phi(t,\xi) = \partial x(t)/\partial x(\xi)$ is the state transition matrix evaluated along the impulsive trajectory. The solution to Eq. (11) in terms of the state transition matrix is

$$\Delta\psi(t) = \Phi'(\xi,t)\Delta\psi(\xi) \tag{13}$$

The difficulty in truncating the series (8) is in the evaluation of Δu. Since $u^{(0)}$ is a series of impulses, Δu can never be "small"; however, from Eq. (12) it is seen that Δu appears only in the integrand and the integral of Δu is indeed small for sufficiently large a_0.

We evaluate Eq. (12) at $t = t_f$, and impose the condition that both the finite-thrust and impulsive trajectories satisfy the same rendezvous boundary conditions. This gives us $\Delta x(0) = 0$ and $\Delta x(t_f) = 0$. Equation (12) becomes

$$\int_0^{t_f} \Phi(t_f,\xi)[u(\xi) - u^{(0)}(\xi)]d\xi = 0 \tag{14}$$

Before proceeding, it is necessary to expand Φ. Assuming that τ_k is small (which is true as $a_0 \to \infty$), then $(\xi - t_k)$ is small and $\Phi(t_k,\xi)$ can be expanded in a Taylor series to give

$$\Phi(t_k,\xi) = I - (\xi - t_k)F_k + \tfrac{1}{2}(\xi - t_k)^2 R_k + \ldots \tag{15}$$

where

$$R_k = \begin{bmatrix} G_k & 0 \\ -\dot{G}_k & G_k \end{bmatrix}$$

Define

$$h(\xi) = [p(\xi)M(\xi)]^{-1} \tag{16}$$

Then Eq. (6) becomes

$$u(\xi) = a_0 h(\xi) M \psi(\xi) \tag{17}$$

As shown in Ref. 8, we also expand $h(\xi)$ in a Taylor series to give

$$h(\xi) = h_k + \dot{h}_k(\xi - t_k) + \tfrac{1}{2}\ddot{h}_k(\xi - t_k)^2 + \dots \tag{18}$$

where h_k, $\dot{h}_k$, and $\ddot{h}_k$ are series in ϵ and δ and are presented in Appendix A through first order and in Ref. 11 through second order.

Using Eqs. (12–17) leads to the following results:

$$\int_{\tau_k} \Phi(t_f,\xi) u^{(0)}(\xi) d\xi = \Delta v_k \Phi_k M \Phi_k' \psi_f^{(0)} \tag{19a}$$

and

$$\int_{\tau_k} \Phi(t_f,\xi) u(\xi) d\xi = a_0 \int_{\tau_k} h(\xi) \Phi(t_f,\xi) M \Phi'(t_f,\xi) d\xi\, \psi_f = \Phi_k I_k \Phi_k' \psi_f \tag{19b}$$

where $\Phi_k = \Phi(t_f,t_k)$. I_k denotes the symmetric matrix

$$I_k = a_0 \int_{\tau_k} \{M h_k + (M\dot{h}_k - N h_k)(\xi - t_k) + (\tfrac{1}{2}M\ddot{h}_k - N\dot{h}_k + Q_k h_k)(\xi - t_k)^2\} d\xi \tag{20}$$

and

$$N = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \qquad Q_k = \begin{bmatrix} I & 0 \\ 0 & G_k \end{bmatrix}$$

$\psi_f^{(0)}$ and ψ_f denote the final values of the adjoint vector for the impulsive and finite-thrust cases, respectively.

Before evaluating I_k, it is necessary to determine the limits of integration. This is done in the next section.

Limits of Integration

The limits of integration for I_k are the switch times t_k^- (thrust-on time) and t_k^+ (thrust-off time). A more convenient set of variables to evaluate these limits can be defined as

$$\tau_k = t_k^+ - t_k^- \tag{21}$$

and

$$\Delta t_k = t_k - t_k^- \tag{22}$$

These times are illustrated in Fig 1. Δt_k locates the beginning of the kth burn with respect to the kth impulse; τ_k is the duration of the kth burn. (Note that for an initial burn $\Delta t_1 = 0$ and for a final burn $\Delta t_f = \tau_f$.) τ_k is expanded as in Eq. (9) and Δt_k is expanded as follows:

$$\Delta t_k = \epsilon \Delta t_k^{(1,0)} + \epsilon\delta \Delta t_k^{(1,1)} + \epsilon^2 \Delta t_k^{(2,0)} + \dots \tag{23}$$

In terms of Δt_k and τ_k, the time-varying terms of Eq. (20) are integrated as follows:

$$\int_{\tau_k} d\xi = \tau_k; \quad \int_{\tau_k} (\xi - t_k) d\xi = \tfrac{1}{2}\tau_k^2 - \tau_k \Delta t_k$$
$$\int_{\tau_k} (\xi - t_k)^2 d\xi = \tfrac{1}{3}\tau_k^3 - \Delta t_k \tau_k^2 + \Delta t_k^2 \tau_k \tag{24}$$

Switch-Point Analysis

The switch function can be written

$$s = p - \alpha \tag{25}$$

where $\alpha = \sigma m / v_j$.

Using Eqs. (3) and (4), it can be shown that

$$\dot{\alpha} = (m_0 a_0 / m v_j)(p - \alpha) \quad \text{thrust on}$$
$$\dot{\alpha} = 0 \quad \text{thrust off} \tag{26}$$

Therefore, α is constant during coast periods and increases monotonically during burns. The change in α across a burn is given by

$$\Delta\alpha_k = \dot{\alpha}(t_k^-)\tau_k + \ddot{\alpha}(t_k^-)(\tau_k^2/2) + \dddot{\alpha}(t_k^-)(\tau_k^3/6) + \dots$$
$$= \Delta\alpha_k^{(0)} + \epsilon\delta\Delta\alpha_k^{(1,1)} + \epsilon^2\delta\Delta\alpha_k^{(2,1)} + \epsilon\delta^2\Delta\alpha_k^{(1,2)} \tag{27}$$

Note that only cross terms in ϵ and δ appear since $\Delta\alpha \to 0$ if either $a_0 \to \infty$ or $v_j \to \infty$.

The value of α at the end of the first burn is chosen equal to unity. Equation (27) can then be used to determine $\alpha(0)$ and thus the initial value of σ. To determine the change in α across an interior burn we must return to the series expressions. α can be represented in a series of the form

$$\alpha = 1 + \epsilon\delta\alpha^{(1,1)} + \epsilon^2\delta\alpha^{(2,1)} + \epsilon\delta^2\alpha^{(1,2)} + \dots \tag{28}$$

Again, this is true because as $a_0 \to \infty$, the impulsive case is approached and as $v_j \to \infty$ the constant thrust-acceleration case is approached. In both cases, α is identically unity for all time.

Substituting in the series for p, m and α and equating terms of like order, it can be shown that, for interior burns,

$$\Delta\alpha_k^{(1,1)} = 0; \quad \Delta\alpha_k^{(2,1)} = -\tfrac{1}{12}\ddot{p}_k^{(0)}\Delta v_k^3$$
$$\Delta\alpha_k^{(1,2)} = 0 \tag{29}$$

The necessary condition that $s \equiv 0$ at switch points is now enforced to each order. This results in a sufficient number of additional conditions to uniquely determine all variables. Following the analysis of Refs. 1 and 8, these conditions for the first burn are

$$(1,0): p_1^{(1,0)} = -\Delta v_1 \dot{p}_1^{(0)} \tag{30a}$$

$$(2,0): p_1^{(2,0)} = -\dot{p}_1^{(0)}\tau_1^{(2,0)} - \tfrac{1}{2}\Delta v_1^2 \ddot{p}_1^{(0)} - \Delta v_1 \dot{p}_1^{(1,0)} \tag{30b}$$

$$(1,1): p_1^{(1,1)} = -\dot{p}_1^{(0)}\tau_1^{(1,1)} \tag{30c}$$

At the beginning of an interior burn,

$$(1,0): p_k^{(1,0)} = 0 \tag{31a}$$

$$(2,0): p_k^{(2,0)} = \dot{p}_k^{(1,0)}\Delta t_k^{(1,0)} - \tfrac{1}{2}\Delta t_k^{(1,0)2}\ddot{p}_k^{(0)} \tag{31b}$$

$$(1,1): p_k^{(1,1)} = 0 \tag{31c}$$

At the end of an interior burn, Eqs. (31a) and (31c) must again hold. Additionally,

$$(2,0): p_k^{(2,0)} = -\dot{p}_k^{(1,0)}(\Delta v_k - \Delta t_k^{(1,0)}) - \tfrac{1}{2}\ddot{p}_k^{(0)}(\Delta v_k^2 - 2\Delta v_k \Delta t_k^{(1,0)} + \Delta t_k^{(1,0)2}) \tag{31d}$$

From Eqs. (31b) and (31d)

$$\Delta t_k^{(1,0)} = \dot{p}_k^{(1,0)}/\ddot{p}_k^{(0)} + \tfrac{1}{2}\Delta v_k \tag{32}$$

For the final burn,

$$p_f^{(1,0)} = \dot{p}_f^{(0)}\Delta v_f \tag{33a}$$

$$p_f^{(2,0)} = \dot{p}_f^{(1,0)}\Delta v_f + \dot{p}_f^{(0)}\tau_f^{(2,0)} - \tfrac{1}{2}\ddot{p}_f^{(0)}\Delta v_f^2 \tag{33b}$$

$$p_f^{(1,1)} = \dot{p}_f^{(0)}\tau_f^{(1,1)} \tag{33c}$$

Equations (30a), (31a) and (33a) represent a total $2n$-2 conditions on $p^{(1,0)}$ or equivalently on $\psi^{(1,0)}$. These $2n$-2 linear algebraic equations will be satisfied by the choice of $\tau^{(2,0)}$ and $\Delta t^{(1,0)}$. Similarly, Eqs. (30b, 31b and 33b) represent $2n$-2 conditions on $p^{(2,0)}$ (or $\psi^{(2,0)}$ (1) and Eqs. (30c, 31c and 33c) represent $2n$-2 conditions on $p^{(1,1)}$ (or $\psi^{(1,1)}$). These conditions will be satisfied by the choice of $\tau^{(3,0)}$, $\Delta t^{(2,0)}$ and $\tau^{(2,1)}$, $\Delta t^{(1,1)}$, respectively.

Enforcement of the switch condition through third order gives the following required results:

$$\Delta t_k^{(2,0)} = \frac{\dot{p}_k^{(2,0)}}{\ddot{p}_k^{(0)}} + \frac{1}{2}\tau_k^{(2,0)} - \frac{\ddot{p}_k^{(1,0)}\dot{p}_k^{(1,0)}}{\ddot{p}_k^{(0)2}} + \frac{\dddot{p}_k^{(0)}}{\ddot{p}_k^{(0)}}\left(\frac{1}{24}\Delta v_k^2 + \frac{1}{2}\frac{\dot{p}_k^{(1,0)2}}{\ddot{p}_k^{(0)2}}\right) \quad (34)$$

and

$$\Delta t_k^{(1,1)} = \dot{p}_k^{(1,1)}/\ddot{p}_k^{(0)} + \tfrac{1}{2}\tau_k^{(1,1)} + \tfrac{1}{12}\Delta v_k^2$$

Evaluation of I_k

The integral in Eq. (14) must be evaluated over the entire trajectory. However, u and $u^{(0)}$ are nonzero only during the thrust intervals τ_k. Therefore, Eq. (14) may be written

$$\sum_{k=1}^{n}\int_{\tau_k}\Phi(t_f,\xi)u(\xi)d\xi - \sum_{k=1}^{n}\int_{\tau_k}\Phi(t_f,\xi)u^{(0)}(\xi)d\xi = 0 \quad (35)$$

Using Eqs. (17) and (7), Eq. (35) can be written in the form

$$W\psi_f - \bar{W}\psi_f^{(0)} = 0 \quad (36)$$

where

$$W = \sum_{k=1}^{n}\Phi_k I_k \Phi_k' \quad (37)$$

and

$$\bar{W} = \sum_{k=1}^{n}\Phi_k M \Phi_k' \Delta v_k \quad (38)$$

I_k can be represented by a series of the form

$$I_k = I_k^{(0)} + \epsilon I_k^{(1,0)} + \delta I_k^{(0,1)} + \epsilon^2 I_k^{(2,0)} + \epsilon\delta I_k^{(1,1)} + \delta^2 I_k^{(0,2)} + \ldots \quad (39)$$

Explicit expressions for the terms $I_k^{(0)}$ and $I_k^{(1,0)}$ are shown in Appendix B, values through second order [$I_k^{(2,0)}$, $I_k^{(1,1)}$, $I_k^{(0,2)}$] are presented in Ref. 11. These expressions are derived by substituting Eqs. (17) and (12) into Eq. (14) and integrating. Define

$$W^{(m,j)} = \sum_{k=1}^{n}\Phi_k I_k^{(m,j)}\Phi_k' \quad (40)$$

Then

$$W = W^{(0)} + \epsilon W^{(1,0)} + \delta W^{(0,1)} + \epsilon^2 W^{(2,0)} + \epsilon\delta W^{(1,1)} + \delta^2 W^{(0,2)} + \ldots \quad (41)$$

Substituting Eq. (41) into Eq. (36) and matching terms of the same order gives

$$(0,0): W^{(0)}\psi_f^{(0)} - \bar{W}\psi_f^{(0)} = 0 \quad (42a)$$

$$(1,0): W^{(1,0)}\psi_f^{(0)} + W^{(0)}\psi_f^{(1,0)} = 0 \quad (42b)$$

$$(0,1): W^{(0,1)}\psi_f^{(0)} = 0 \quad (43c)$$

$$(2,0): W^{(2,0)}\psi_f^{(0)} + W^{(1,0)}\psi_f^{(1,0)} + W^{(0)}\psi_f^{(2,0)} = 0 \quad (42d)$$

$$(1,1): W^{(1,1)}\psi_f^{(0)} + W^{(0,1)}\psi_f^{(1,0)} + W^{(0)}\psi_f^{(1,1)} = 0 \quad (42e)$$

$$(0,2): W^{(0,2)}\psi_f^{(0)} = 0 \quad (42f)$$

Equation (42a) is satisfied by taking $\tau_k^{(1,0)} = \Delta v_k$, thus making $W^{(0)} = \bar{W}$. Using the results of the switch function, each equation of (42) becomes a linear function of the highest order coefficient of the burn time expansion $\tau_k^{(m+1,j)}$. As a result, Eqs. (42) may be used to solve for the coefficients of the burn time expansion recursively. In principle, the recursive relationship may be extended to any order, although the equations become algebraically intractable above second order.

First-Order Solution

The equation which determines the first-order corrections $\psi^{(1,0)}$ and $\tau_k^{(2,0)}$ is Eq. (42a),

$$W^{(1,0)}\psi_f^{(0)} + W^{(0)}\psi_f^{(1,0)} = 0$$

This cannot be solved simply by inverting $W^{(1,0)}$ and solving for $\psi^{(1,0)}$ since $W^{(1,0)}$ is a function of both $\psi^{(1,0)}$ and $\tau_k^{(2,0)}$—the latter is yet undetermined. The product $W^{(1,0)}\psi^{(0)}$ can be written

$$W^{(1,0)}\psi_f^{(0)} = A\psi_f^{(1,0)} + B^{(1,0)}\psi_f^{(0)} + C^{(1,0)}\psi_f^{(0)} \quad (43)$$

where

$$A = -\sum_{k=2}^{n-1}[\Delta v_k/\ddot{p}_k^{(0)}]e_k e_k'$$

$$B^{(1,0)} = \sum_{k=1,n}\Phi_k[M\dot{p}_k^{(0)} - N]\Phi_k' \left[\frac{1}{2}\tau_k^{(1,0)2} - \tau_k^{(1,0)}\Delta t_k^{(1,0)}\right]$$

$$C^{(1,0)} = \sum_{k=1}^{n}\Phi_k M \Phi_k' \tau_k^{(2,0)}$$

and

$$e_k = \Phi_k N \psi_k^{(0)}$$

Note that the summation in A is only over interior thrusts and the summation in $B^{(1,0)}$ is only over terminal thrusts.

The solution to Eq (42a) is

$$\psi_f^{(1,0)} = -[W^{(0)} + A]^{-1}[B^{(1,0)} + C^{(1,0)}]\psi_f^{(0)} \quad (44)$$

By its definition [see Eq. (38)] $W^{(0)}$ is a positive semidefinite symmetric matrix; it can be shown that it must be nonsingular (i.e., positive definite) if the optimal control is unique. A is also semidefinite [note: $p(0) < 0$ at interior impulses]. Therefore, $[W^{(0)} + A]^{-1}$ exists if the optimal impulsive control is unique.

The one complication in the solution (45) is that $C^{(1,0)}$ is not yet known, since it depends upon $\tau_k^{(2,0)}$. This is resolved in the following manner. The third term in Eq. (43) can be rewritten

$$C^{(1,0)}\psi_f^{(0)} = Uc^{(1,0)}$$

where

$$U = [\Phi_1 M\psi_1^{(0)} \vdots \Phi_2 M\psi_2^{(0)} \vdots \ldots \vdots M\psi_f^{(0)}]$$

and

$$c^{(1,0)\prime} = [\tau_1^{(2,0)},\ldots,\tau_f^{(2,0)}]$$

In addition, define the vector

$$l^{(1,0)} = \begin{pmatrix} p_1^{(1,0)} \\ \vdots \\ p_f^{(1,0)} \end{pmatrix} = U'\psi_f^{(1,0)}$$

From the switch point analysis (30a), (31a) and (33a), $l^{(1,0)}$ is known.

Premultiplying Eq. (45) by U' and rearranging

$$U'[W^{(0)} + A]^{-1}Uc^{(1,0)} = b^{(1,0)} - l^{(1,0)} \quad (45)$$

where

$$b^{(1,0)} = -U'[W^{(0)} + A]^{-1}B^{(1,0)}\psi_f^{(0)}$$

is a known vector.

In Ref. 9, Potter and Stern show that the columns of U are linearly independent if the optimal trajectory has the minimum number of impulses. Assuming this to be the case, the matrix on the left-hand side of Eq. (46) is invertible and

$$c^{(1,0)} = \{U'[W^{(0)} + A]^{-1}U\}^{-1}[b^{(1,0)} - l^{(1,0)}] \quad (46)$$

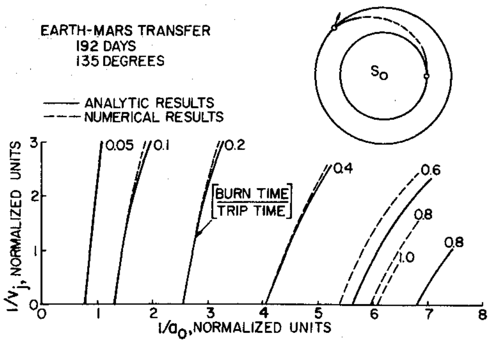

Fig. 2 Comparison of analytic and numerical burn time data for a two-burn transfer.

Therefore, the $\tau_k^{(2,0)}$ are determined from Eq. (47) and then $\psi^{(1,0)}$ is determined from Eq. (44). The preceding criteria on the invertibility of the matrices $[W^{(0)} + A]$ and $\{U'[W^{(0)} + A]^{-1}U\}$ are sufficient conditions for an impulsive trajectory to be the limit case of a family of finite-thrust trajectories satisfying the same boundary conditions. Because

$$\lim_{a_0 \to \infty} \psi = \psi^{(0)}$$

independent of the value of v_j, all coefficients of the form $\psi^{(0,m)}$ $m = 1,2,3 \ldots$, are zero. As a result, only one term appears in Eq. (42c). $\psi_f^{(0)}$ is not equal to zero, thus $W^{(0,1)}$ must be. To insure that

$$W^{(0,1)} = \sum_{k=1}^{n} \Phi_k I_k^{(0,1)} \Phi_k' = 0$$

set $I_k^{(0,1)} = 0$, $k = 1,2,\ldots n$. This condition provides a direct solution for $\tau_k^{(1,1)}$:

$$\tau_k^{(1,1)} = -\left(\sum_{i=1}^{k=1} (\Delta v_i + \frac{1}{2} \Delta v_k)\right) \Delta v_k \qquad (47)$$

Note also that $\Delta t_k^{(0,1)} = 0$ because of the above limit.

Second-Order Solution

The set of equations to be solved for the second-order terms are Eqs. (42d-f). Equation (42d) is solved for $\tau^{(3,0)}$ and $\psi^{(2,0)}$, Eq. (42e) is solved for $\tau^{(2,1)}$ and $\psi^{(1,1)}$, and Eq. (42f) is solved for $\tau^{(1,2)}$ and $\psi^{(0,2)}$. Details of these solutions are presented in Ref. 11.

Numerical Results

The accuracy of the analytic solution derived herein is dependent on the ratio of total Δv to a_0 and the "linearity" of the gravitational field in the neighborhood of the impulsive trajectory. The units of $\Delta v/a_0$ are time and the unitless characteristic time is the ratio of burn time to total trip time. Clearly, as $a_0 \to \infty$, regardless of the value of v_j, the characteristic time approaches zero and the solution becomes exact. However, accurate results are obtainable for burn times of 30 to 50% of the total trip time and for values of v_j commensurate with typical propulsion systems. The solution is, thus, applicable to chemical, nuclear and certain forms of electric propulsion.

Two example cases are presented, both corresponding to orbital transfers between circular, coplanar orbits of radii 1.0 and 1.52 a.u. The first case is a 135° transfer, approximately equivalent to a 192-day Earth-Mars rendezvous. This trajectory has two burns and is summarized in Table 1 and Fig. 2. Accurate prediction of the burn time is obtained for burn times totalling up to 50% of the total transfer time. In these examples, accurate numerical solutions for comparison have been generated by a calculus of variations optimization computer program.[10] The analytic data generated for this case is complete through second order and required about ½ sec to generate on a CDC 6400 computer.

The second example is a 220° Earth-Mars rendezvous transfer with a trip time of 259 days. This trajectory, shown in Fig. 3, uses three burns. Trajectory data for this case is presented in Table 2 for first-order terms only. The time required to compute the analytic trajectory data for this case was about 2 sec, the majority of which was required to obtain the optimal three-impulse reference solution. It can be seen that the first-order data alone gives a reasonably accurate solution for burn time up to approximately 30% of the total trip time.

Summary

An analytic solution for constant-thrust, minimum-propellant space trajectories has been developed in terms of a two-variable expansion about the minimum-Δv, impulsive trajectory. The solution, a series in the inverse of the thrust-acceleration, $\epsilon = 1/a_0$, and the rocket jet velocity, $\delta = 1/v_j$, requires only the solution of a set of algebraic equations—no numerical integration or iteration is necessary. The solution provides the trajectory-state and adjoint-time histories and requires about two orders of magnitude less computing time than comparable numerical integration optimization programs. In addition, one analytic solution is applicable to a wide range of propulsion parameters corresponding to many different vehicles as opposed to numerical programs which

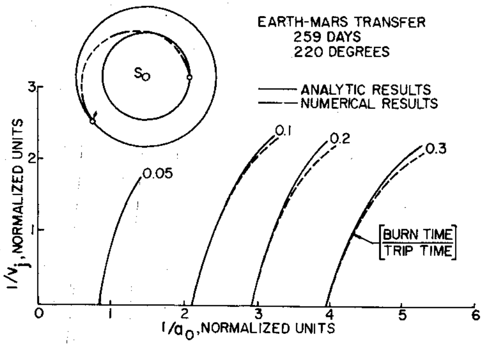

Fig. 3 Comparison of analytic and numerical burn time data for a three-burn transfer.

Table 1 Earth-Mars transfer circular, coplanar orbits 192 days, 135°

Initial state	Final state	Impulses
$x_0 = \begin{bmatrix} 1.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 1.0 \\ 0.0 \end{bmatrix}$	$x_f = \begin{bmatrix} -1.06066 \\ 1.06066 \\ 0.0 \\ -0.57735 \\ -0.57735 \\ 0.0 \end{bmatrix}$	$\Delta v_1 = 0.12712$ $\Delta v_2 = 0.10875$
Impulsive adjoint		
$\psi_1^{(0)} = \begin{bmatrix} 1.01023 \\ 0.53969 \\ 0.0 \\ 0.68563 \\ 0.72795 \\ 0.0 \end{bmatrix}$	$\psi_f^{(0)} = \begin{bmatrix} -0.09224 \\ 0.68069 \\ 0.0 \\ -0.15900 \\ -0.98727 \\ 0.0 \end{bmatrix}$	
First-order terms		
$\psi_1^{(1,0)} = \begin{bmatrix} 0.1326 \\ 0.0937 \\ 0.0 \\ 0.1289 \\ 0.0682 \\ 0.0 \end{bmatrix}$	$c^{(2,0)} = \begin{bmatrix} 0.00854 \\ 0.00411 \end{bmatrix}$	$c^{(1,1)} = \begin{bmatrix} -0.00808 \\ -0.01973 \end{bmatrix}$
Second-order terms		
$\psi_1^{(2,0)} = \begin{bmatrix} 0.0278 \\ 0.0118 \\ 0.0 \\ 0.0180 \\ 0.0205 \\ 0.0 \end{bmatrix}$	$\psi_1^{(2,1)} = \begin{bmatrix} 0.00673 \\ 0.00728 \\ 0.0 \\ 0.00726 \\ 0.00521 \\ 0.0 \end{bmatrix}$	$c^{(3,0)} = \begin{bmatrix} 0.00107 \\ 0.00036 \end{bmatrix}$ $c^{(2,1)} = \begin{bmatrix} -0.00154 \\ -0.00258 \end{bmatrix}$ $c^{(1,2)} = \begin{bmatrix} 0.00034 \\ 0.00184 \end{bmatrix}$

Table 2 Earth-Mars transfer circular, coplanar orbits 259 days, 220°

Initial state	Midcourse state	Final state	Impulses
$x_0 = \begin{bmatrix} 1.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 1.0 \\ 0.0 \end{bmatrix}$	$x_m^- = \begin{bmatrix} 0.3939 \\ 0.8900 \\ 0.0 \\ -0.9435 \\ 0.4008 \\ 0.0 \end{bmatrix}^a$	$x_f = \begin{bmatrix} -1.1769 \\ -0.9676 \\ 0.0 \\ 0.5145 \\ -0.6258 \\ 0.0 \end{bmatrix}$	$\Delta v_1 = 0.0269$ $\Delta v_2 = 0.1100$ $(t_2 = 1.1235)$ $\Delta v_3 = 0.1610$
Impulsive adjoint			
$\psi_1^{(0)} = \begin{bmatrix} -0.6781 \\ -0.7164 \\ 0.0 \\ -0.9959 \\ -0.0903 \\ 0.0 \end{bmatrix}$	$\psi_m^{(0)} = \begin{bmatrix} 0.0768 \\ 0.3715 \\ 0.0 \\ -0.9793 \\ 0.2024 \\ 0.0 \end{bmatrix}$	$\psi_f^{(0)} = \begin{bmatrix} -0.9864 \\ -0.3946 \\ 0.0 \\ 0.9929 \\ 0.1188 \\ 0.0 \end{bmatrix}$	
First-order terms			
$\psi_1^{(1,0)} = \begin{bmatrix} -0.0237 \\ -0.0045 \\ 0.0 \\ -0.0231 \\ -0.0344 \\ 0.0 \end{bmatrix}$	$c^{(2,0)} = \begin{bmatrix} 0.0100 \\ 0.0019 \\ 0.0043 \end{bmatrix}$	$c^{(1,1)} = \begin{bmatrix} 0.00362 \\ 0.00677 \\ 0.01563 \end{bmatrix}$	

[a] Before the midcourse impulse.

must solve for the optimal trajectory for each set of vehicle parameters separately.

The solution technique used is quite general and works equally well for other boundary condition types, e.g., flyby or optimum transfer angle trajectories. In each case, specification of the trajectory type leads to boundary conditions on the state or adjoint which must be satisfied by simultaneous solution of the resulting algebraic equations of state and adjoint at each order in a_0 and v_j. Also, propulsion profiles other than constant-thrust, constant-jet velocity can be dealt with by the same solution technique. The results obtained can be used directly or can provide an accurate starting guess for a precision, iterating, numerical optimization computer program.

Appendix A: Expansion of h

$$h_k = 1 - \frac{1}{a_0} p_k^{(1,0)} + \frac{1}{v_j}\left(\sum_{i=1}^{k-1} \Delta v_i + \Delta t_k^{(1,0)}\right) + \ldots$$

$$\dot{h}_k = -\dot{p}_k^{(0)} + \frac{a_0}{v_j} + \ldots$$

$$\ddot{h}_k = 0 + \ldots$$

Appendix B: Expansion of I_k

$$I_k^{(0)} = M\tau_k^{(1,0)} + \ldots$$

$$I_k^{(1,0)} = M[\tau_k^{(2,0)} + \tau_k^{(1,0)}h_k^{(1,0)}] + [M\dot{h}_k^{(0)} - N][\tfrac{1}{2}\tau_k^{(1,0)2} - \tau_k^{(1,0)}\Delta t_k^{(1,0)}] + \ldots$$

$$I_k^{(0,1)} = M\{\tau_k^{(1,1)} + h_k^{(0,1)}\tau_k^{(1,0)} + \dot{h}_k^{(-1,1)}[\tfrac{1}{2}\tau_k^{(1,0)2} - \tau_k^{(1,0)}\Delta t_k^{(1,0)}]\} + \ldots$$

References

[1] Hazelrigg, G. A. and Lion, P. M., "Analytical Determination of the Adjoint Vector for Optimum Space Trajectories," AIAA Paper 69-916, Princeton, N. J., 1969.

[2] Lawden, D. F., *Optimal Trajectories for Space Navigation*, Butterworths, London, 1962.

[3] Neustadt, L. W., "A General Theory of Minimum Fuel Space Trajectories," *SIAM Journal on Control*, Vol. 3, No. 2, 1965, pp. 317–356.

[4] Pines, S., "Constants of the Motion for Optimal Thrust Trajectories in a Central Force Field," *AIAA Journal*, Vol. 2, No. 11, Nov. 1964, pp. 2010–2014.

[5] Handelsman, M., "Optimal Free-Space Fixed-Thrust Trajectories Using Impulsive Trajectories as Starting Iteratives," *AIAA Journal*, Vol. 4, No. 6, June 1966, pp. 1077–1082.

[6] Lion, P. M. and Handelsman, M., "The Primer Vector on Fixed-Time Impulsive Trajectories," *AIAA Journal*, Vol. 6, No. 1, Jan. 1968, pp. 127–132.

[7] Minkoff, M. and Lion, P. M., "Optimal Multi-Impulse Rendezvous Trajectories," Paper AD 143, XIX IAF Congress, New York, Oct. 1968.

[8] Hazelrigg, G. A., "An Analytic Study of Multi-Burn Space Trajectories," Ph.D. thesis, Sept. 1968, Princeton Univ., Princeton, N. J.

[9] Potter, J. E. and Stern, R. E., "Optimization of Midcourse Velocity Corrections," *Proceedings of the IFAC Symposium*, Stavaugh, Norway, Plenum Press, New York, 1965.

[10] Lion, P. M., Campbell, J. H., and Shulzycki, A. B., "TOPCAT—Trajectory Optimization Program for Comparing Advanced Technologies," Aerospace Engineering Rept. 717s, March 1968, Princeton Univ., Princeton, N. J.

[11] Hazelrigg, G. A., Kornhauser, A. L., and Lion, P. M., "An Analytic Solution for Constant-Thrust, Optimal-Coast, Minimum-Propellant Space Trajectories," Aerospace Engineering Rept. 916d, 1970, Princeton Univ., Princeton, N. J.

[12] Robbins, H. M., "An Analytical Study of the Impulsive Approximation," *AIAA Journal*, Vol. 4, No. 8, Aug. 1966, pp. 1417–1423.